Special Relativity 3

Length Contraction

1. Now put two observers on either end of the ship and one on the ground.
2. The observers on the ship each note the time that they pass the observer on the ground and they calculate t_s as the difference between the two.
3. The observer on the ground notes the time between the passage of the two ends of the ship and calls that t_g.
4. The "events" here are the times that each end of the ship passed the observer on the ground. So that observer sees the proper time: $t_g = t_0$.
5. Hence $t_s = \frac{t_g}{\sqrt{1-\frac{v^2}{c^2}}}$.
6. The observers on the ship calculate $L_s = v \times t_s$ an the one on the ground calculates $L_g = v \times t_g$.
7. You might think that L_g would be the "proper" length because it was calculated with the proper time, but no. The *proper length* is defined as the length of the object measured in *that object's own frame of reference* (which makes sense when you think about it, since this is the only way of defining it uniquely).
8. We want to find the multiplier which will take us from the proper lentgh $L_0 = L_s$ to the other length L_g.
9. So we want: $v\frac{t_g}{\sqrt{1-\frac{v^2}{c^2}}} \times (?) = vt_g$
10. Cancel vt_g from both sides and it's obvious that we have to multiply by $\sqrt{1-\frac{v^2}{c^2}}$.
11. So: $L = L_0\sqrt{1-\frac{v^2}{c^2}}$.

Summary

1. The *proper time between two events*, t_0, is the time measured by an observer who is present at both events.
2. The *proper length of an object*, L_0, is the length as measured in the object's own frame of reference.
3. $t = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}}$.
4. $L = L_0\sqrt{1-\frac{v^2}{c^2}}$.

Invariance and the spacetime "interval"

I'm going to shift gears now for a bit, and talk about something called invariance. Invariance is the property of not changing when subjected to some specific set of operations.

As a first example, consider a vector in 2 dimensional euclian space (an arrow on the whiteboard). If I draw an xy coordinate system, then I can represent the arrow as a column vector. Perhaps the head of the arrow is at the point x=3 and y=2. Then the components of the column vector will be 3 and 2.

If I change the coordinate system (rotate the xy axes by a few degrees) then the components of the column vector will change. The components of a vector are not invariant. But the length of the vector will never change no matter how much I rotate the axes. The length doesn't *vary* and so it is called invariant.

We would say something like: *the length of a vector in euclidean space is invariant under rotations of the coordinate system.*

This is more than a mathematical nicety. The implication is that the length of a vector is a *real* thing, whereas the components in the vector's column representation are merely *artifacts* of some specific choice of coordinates.

Similarly, in a complex vector space (the space of quantum bits) the length of a vector $|v\rangle$ is $\sqrt{\langle v|v\rangle}$. $|v\rangle$ can be expressed in any number of different bases, but no matter what basis we select the length will not change. Change of basis is a little bit more general than simple rotation, but a vector's length is still invariant under change of basis. A change of basis is also a change of coordinate systems.

Now back to special relativity. Before Einstein, we thought that we lived in 3 dimensional euclidean space. But we have now seen that lengths in space (and in time!) *change* based on the reference frame of the observer. In other words:

1. A particular observer's frame of reference is a coordinate system.
2. When we change from one observer's viewpoint to another we are changing the coordinate system (like a change of basis).
3. Extensions in space and time are merely artifacts of the specific coordinate system chosen.
4. So, we can conclude that we don't actually live in 3D euclidean space!

If the extensions in time and space are not "real" then what is real? It turns out that if you take all the space extensions and add them up as if you were going to calculate the distance in euclidean space, and then *subtract* the time extension, you will get an invariant quantitiy:

$$s = \sqrt{x^2 + y^2 + z^2 - t^2}$$

If we don't live in euclidean space the where do we live? Not long after Einstein published his first paper on what we now know as special relativity, Hermann Minkowski observed that the theory had a startling implication. That we actually live in a four dimensional world of space-time (often called Minkowski spacetime or simply Minkowski space).

The invariant s above is called the *spacetime interval* and you could (perhaps) think of it as the true *distance* in four dimensional spacetime.

It is very important to understand the difference between spacetime and 4D euclidean space. Suppose we simply had a "fourth dimension" of space. This would mean that we would be able to find *four* angles that were all mutually orthogonal (all 90° from each other). If we called the fourth direction w then the distance in this space would be given by the formula:

$$D = \sqrt{x^2 + y^2 + z^2 + w^2}$$

The minus sign in the formula for the space time interval makes it clear that time is not just "the fourth dimension" that is portrayed in sci-fi movies. Time is defintely not at a 90° angle from space. It's important to keep this in mind, since almost every diagram of spacetime (including the ones that we are going to draw) make it *appear* as if time *is* at a 90° angle.

What "direction" *does* time go? I have no idea.