Vector Spaces Axioms

Here are the vector space "axioms." Any mathematical entity which follows these rules can be thought of as a vector space. Note that any letters that are not inside kets represent scalars (just regular numbers).

1. Closure with regard to vector addition: $|v_1\rangle + |v_2\rangle = |v_3\rangle$.
   (In other words, if you add two vectors you get a vector.)
2. Closure with regard to scalar multiplication: $a|v_1\rangle = |v_2\rangle$.
3. Additive identity vector element: There exists a vector $\mathbf{0}$ such that $|v\rangle + \mathbf{0} = |v\rangle$.
4. Additive inverse elements: For every vector $|v\rangle$ there exists a vector $|-v\rangle$ such that $|v\rangle + |-v\rangle = \mathbf{0}$.
5. Multiplicative identity scalar element: There exists a scalar 1 such that $1|v\rangle = |v\rangle$.
6. Vector addition is commutative: $|v_1\rangle + |v_2\rangle = |v_2\rangle + |v_1\rangle$.
7. Vector addition is associative: $|v_1\rangle + (|v_2\rangle + |v_3\rangle) = (|v_1\rangle + |v_2\rangle) + |v_3\rangle$
8. Scalar multiplication is commutative: $a|v\rangle = |v\rangle a$.
9. Scalar multiplication is associative: $a(b|v\rangle) = (ab)|v\rangle$.
10. Vectors distribute over scalar addition: $|v\rangle(a + b) = a|v\rangle + b|v\rangle$.
11. Scalar multiplication distributes over vector addition: $a\left(|v_1\rangle + |v_2\rangle\right) = a|v_1\rangle + a|v_2\rangle$.

In axioms 3 and 4, why did I use the symbol $\mathbf{0}$ instead of something like $|0\rangle$? The reason is that the expression $|0\rangle$ typically refers to the a bit with the value of a "classical zero." This is actually the vector with a one and a zero in it. But the zero vector is a vector with all zeros in it:

$$\text{classical zero} = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ classical one} = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ the zero vector} = \mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Notice that although you can multiply a vector by a scalar, *there is no rule for muliplying two vectors together.* Various different kinds of vector products can be defined, but they are not part of the definition of a vector space.